Marking scheme Exam CFD Jan 29, 2015

Problem 1

a) $$C\phi_j = \frac{\phi_{j+1} - \phi_{j-1}}{2h} - \lambda \frac{\phi_{j+1} - 3\phi_j + 3\phi_{j-1} - \phi_{j-2}}{h}$$

Taylor series at x_j :

$$\phi_{j+1} = \phi_j + h\phi_j' + \tfrac{1}{2}h^2\phi_j'' + \tfrac{1}{6}h^3\phi_j''' + \mathcal{O}(h^4)$$

$$\phi_{j-1} = \phi_j - h\phi_j' + \tfrac{1}{2}h^2\phi_j'' - \tfrac{1}{6}h^3\phi_j''' + \mathcal{O}(h^4)$$

1pt $$\phi_{j-2} = \phi_j - 2h\phi_j' + 2h^2\phi_j'' - \tfrac{8}{6}h^3\phi_j''' + \mathcal{O}(h^4)$$

Substitution gives

1pt $$C\phi_j = \phi_j' + \left(\tfrac{1}{6} - \lambda\right)h^2\phi_j''' + \mathcal{O}(h^4)$$

So, $C\phi_j$ provides a second-order approximation

1pt of ϕ_j', and is third-order accurate if $\lambda = \frac{1}{6}$

b) $C\phi_j = 0$

1pt try $\phi_j = r^j$

substitution yields

$$\left(\tfrac{1}{2} - \lambda\right)r^3 + 3\lambda r^2 - \left(\tfrac{1}{2} + 3\lambda\right)r + \lambda = 0$$

$r = 1$ is a solution

$$(r-1)\left[\left(\tfrac{1}{2} - \lambda\right)r^2 + \left(\tfrac{1}{2} + 2\lambda\right)r - \lambda\right] = 0$$

$$r = 1 \ \vee \ r_\pm = \frac{-\left(\tfrac{1}{2} + 2\lambda\right) \pm \sqrt{\tfrac{1}{4} + 4\lambda}}{1 - 2\lambda}$$

General solution

1pt $\phi_j = c_0 + c_+ r_+^j + c_- r_-^j$

$\lambda \to 0 \quad r_+ \to 0 \quad r_- \to -1$

1pt $\phi_j \to c_0 + c_-(-1)^j$

c)

c_0, c_+ and c_- are to be set with the

1pt help of the boundary condition

Taking the boundary at $j=0$ gives

$\phi_0 = 1$

1pt linear extrapolation, for example, yields

$\phi_{-1} = 2\phi_0 - \phi_1$

$\phi_{-2} = 2\phi_0 - \phi_2$

d) $C\phi$ in matrix notation:

$$\underbrace{\begin{pmatrix} \ddots & & & \\ \frac{\lambda}{h} & -\frac{3\lambda}{h} - \frac{1}{2h} & \frac{3\lambda}{h} & -\frac{\lambda}{h} + \frac{1}{2h} \\ & & & \ddots \end{pmatrix}}_{A} \begin{pmatrix} \phi_{j-2} \\ \phi_{j-1} \\ \phi_j \\ \phi_{j+1} \end{pmatrix}$$

1pt

Symmetric part $\frac{1}{2}(A + A^*)$

Skew sym part $\frac{1}{2}(A - A^*)$

The artificial dissipation is given by the

1pt symmetric part $\frac{1}{2}(A+A^*)\,\phi$

Problem 2

a) Forward Euler :

1pt $$\frac{\phi_j^{n+1}-\phi_j^n}{\delta t} + u\,\frac{\phi_{j+1}^n-\phi_{j-1}^n}{2h} - k\,\frac{\phi_{j+1}^n-2\phi_j^n+\phi_{j-1}^n}{h^2} = f_j^n$$

or

$$\phi_j^{n+1} = C_-\,\phi_{j-1}^n + C_0\,\phi_j^n + C_+\,\phi_{j+1}^n + \delta t\, f_j^n$$

with

$$C_- = \frac{u\delta t}{2h} + \frac{k\delta t}{h^2}$$

$$C_0 = 1 - \frac{2k\delta t}{h^2}$$

$$C_+ = -\frac{u\delta t}{2h} + \frac{k\delta t}{h^2}$$

$$C_- + C_0 + C_+ = 1$$

1pt positive operator if $C_-, C_0, C_+ \geq 0$

$C_0 > 0$ yields $\delta t \leq h^2/2k$

1pt $C_\pm > 0$ yields $P = \dfrac{|u|h}{k} \leq 2$

b) Fourier amplification factor

1pt substitute $\phi_j^n = c_\theta^n\, e^{ij\theta}$

this gives

$$c^{n+1} = a(\theta)\, c^n$$

amplification factor

1pt abs. stability condition $|g(\theta)| \le 1$

Here

$$g(\theta) = 2\frac{\delta t k}{h^2}(\cos\theta - 1) + 1$$

1pt $$+ \; i\,\frac{\delta t u}{h}$$

Problem 3

a) $L\phi = (u\phi)_x + (v\phi)_y \qquad u_x + v_y = 0$

$$\int_\Omega \psi L\phi \, d\Omega = \iint_\Omega \psi\left[(u\phi)_x + (v\phi)_y\right] dx\,dy =$$

1pt integration by parts

$$= - \iint_\Omega \psi_x u\phi + \psi_y v\phi \; dx\,dy + \text{boundary term}$$

1pt $$= - \iint (u\psi)_x \phi + (v\psi)_y \phi \; dx\,dy + \text{boundary term}$$

because $u_x + v_y = 0$

b)

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \phi^2 \, d\Omega = \frac{1}{2}\int_\Omega (\phi L\phi + L\phi\,\phi)\, d\Omega$$

1pt $$= \int_\Omega \phi \; \tfrac{1}{2}(L + L^*)\,\phi \; dx = 0$$

if $L + L^* = 0$

c) $$(L\phi)_{ij} = \frac{(u\phi)_{i+\frac{1}{2}j} - (u\phi)_{i-\frac{1}{2}j}}{\delta x}$$

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1pt $$+ \frac{(v\phi)_{ij+\frac{1}{2}} - (v\phi)_{ij-\frac{1}{2}}}{\delta y}$$

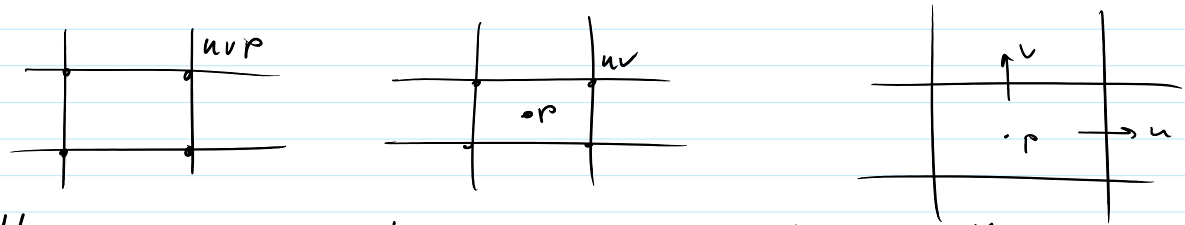
with
$$(u\phi)_{i+\frac{1}{2}j} = \frac{1}{2}\left(u_{i+1j} + u_{ij}\right) \cdot \frac{1}{2}\left(\phi_{i+1j} + \phi_{ij}\right)$$

2pt $$(v\phi)_{ij+\frac{1}{2}} = \frac{1}{2}\left(v_{ij+1} + v_{ij}\right) \frac{1}{2}\left(\phi_{ij+1} + \phi_{ij}\right)$$

Problem 4

a)

2pt



cell corners — A

velo corners, pressure centre — B

MAC — C

discrete pressure gradient is denoted by $G p$

A: $\dim(\ker G) = 4$

1pt B: $\dim(\ker G) = 2$

C: $\dim(\ker G) = 0$

b) MAC discretization of Navier-Stokes is of the form

1pt $$\Omega \frac{u^{n+1} - u^n}{\delta t} + G p^{n+1} = R^n$$

discretization of incompressibility constraint:

1pt $$D u^{n+1} = 0$$

Relation $G^* = -D$

Poisson eq. for the pressure becomes

1pt $$D\Omega^{-1}G\,p^{n+1} = D\Omega^{-1}R^n + \frac{1}{\delta t}Du^n$$

c) Boundary conditions for the (normal) velocity are directly substituted into

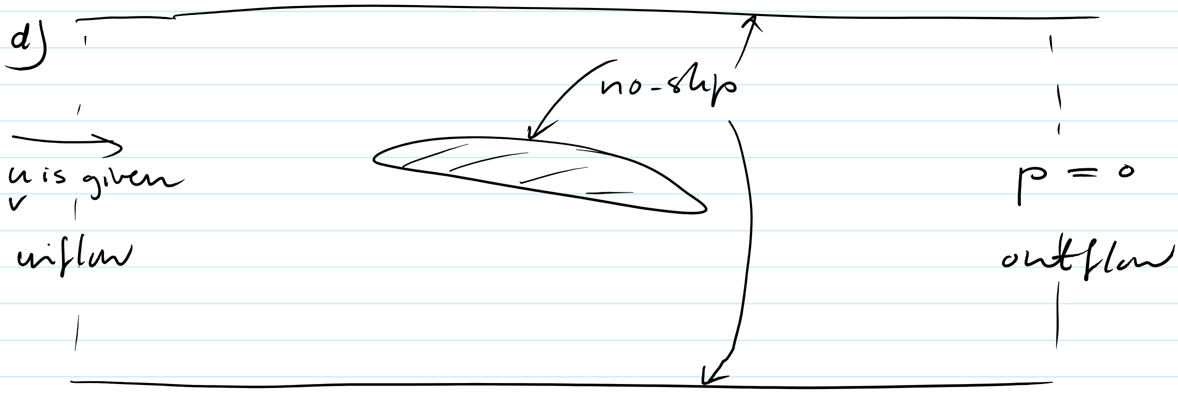
1pt $$Du^{n+1} = 0$$

Then $D\Omega^{-1}G$ const vector $= 0$

Hence $p =$ constant is the eigenvector

1pt associated with the eigenvalue 0.

d)

2pt 

Problem 5

a) $l_K = Re^{-3/4}\, l_{ed} \Rightarrow dx, dy, dz = \mathcal{O}(Re^{-3/4})$

$\tau_K = Re^{-1/2}\, \tau_{ed} \Rightarrow dt = \mathcal{O}(Re^{-1/2})$

number of gridpoint $\propto (Re^{3/4})^3$

number of timesteps $\propto Re^{1/2}$

complexity $\propto (Re^{3/4})^3\, Re^{1/2} = Re^{11/4}$

2pt $Re \to 10 * Re$: complexity $\to 10^{11/4}$ complex.

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b) Compared are

1pt time average velocities and

1pt root-mean-square velocities, e.g.